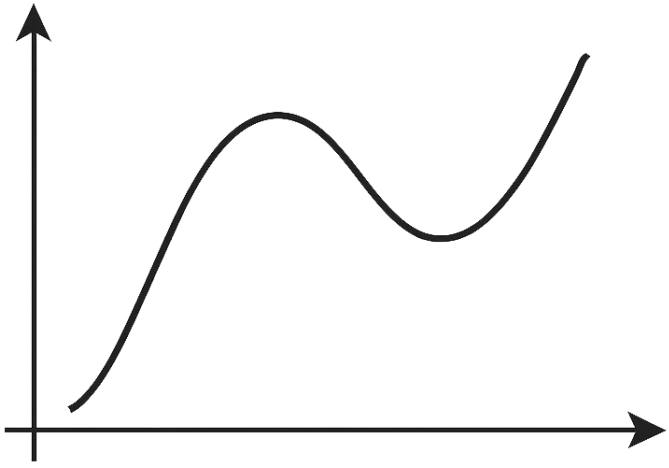
Calculus On Tour



Mathematics Gives Direction
Ethical Reasoning Guides the Path

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Welcome to Calculus on Tour!

This tour invites you to explore calculus not just as a mathematical tool, but as a way of thinking about the world. Behind every function, graph, and optimization problem lies a real-life context – decisions that shape individuals, communities, and the future we're building together. Each stop on the tour introduces key concepts in calculus alongside meaningful challenges drawn from fields like economics, health, technology, and sustainability. You'll model systems, analyze trade-offs, and reflect on how ethical reasoning can guide responsible action. Whether you're solving for limits, sketching curves, or evaluating policy through derivatives, remember: mathematics gives us direction – ethical reasoning guides the path.

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- **T**hink Critically – Reason ethically: evaluate trade-offs, justify decisions, consider broader consequences.

These three components (A, C, and T) can occur in different orders and iterations. Sometimes, you need to Think Critically (T) before you Analyze (A)!

Imagine a famous artist is coming to town. Tickets for similar concerts usually go for around \$100. If someone offered you one for \$60, you'd probably grab it right away. At \$120, it still feels worth it to many. But if the price jumps to \$300 or more? Most people would hesitate, unless it came with a backstage pass.

This illustrates how demand changes with price – what we call a **price response function**. Some fans are willing to pay extra for the experience; others only go if it's affordable. For most people, willingness to pay drops sharply as prices double, triple, or more.

These kinds of functions help us model real-world behavior wherever price influences choices, such as retail pricing, airline tickets, app subscriptions, housing, and many more.

A common way to model this is with a **logistic function**

$$f(x) = \frac{100}{1 + e^{0.1(x-200)}},$$

where x is the price (in dollars) and $f(x)$ is the demand (in thousands of tickets). The parameter 0.1 controls how sensitive demand is to price changes. Another closely related model is the **logit function**

$$g(x) = \frac{100e^{0.1(200-x)}}{1 + e^{0.1(200-x)}}.$$

1. [A+C] What do you observe about f and g?

2. [A+C] What is limit of f as x approaches 60, 100, 200, and 300? What do you observe?

3. [A] What is limit of f as x approaches ∞?

4. [A] In reality, price is typically nonnegative. But if we consider the function by itself, what is limit of f as x approaches $-\infty$?

5. [T] **Ethical Reasoning**

   Global average ticket price rose 23.3% to \$130.81 in 2023[1]. Some tour tickets frequently resold for hundreds or even thousands of dollars.

   (a) At \$1,000+ per ticket, only a few can afford to attend top-tier concerts. Who is excluded by such pricing, and why does that matter?

   (b) Companies like Ticketmaster use algorithms to adjust prices in real time. Is it fair to charge fans more precisely because they show interest?

   (c) High prices benefit artists and promoters, but may be unaffordable for many. With major platforms dominating ticket sales, fans often lack pricing transparency. What ethical obligations do ticket platforms owe to fans and artists?

   (d) You're encouraged to reflect using frameworks like utilitarianism (maximizing enjoyment for the most fans), deontology (duty to fairness), or virtue ethics (acts of generosity or respect for fans).

[1] https://www.theguardian.com/culture/2025/jan/20/gig-concert-ticket-prices-dynamic-pricing-oasis-taylor-swift-eras

2 Calculus on Tour: Continuity – No Sudden Jumps

ACT for Ethical Reasoning

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An electronics company is testing a sensitive circuit component that regulates voltage. The system is designed to handle up to 3 milliamps (mA). As the current increases and approaches 3 mA, the voltage regulator becomes unstable. At exactly 3 mA, the system shuts down, producing no output. For currents above 3 mA, a secondary circuit takes over with a new response pattern. The system is modeled by

$$V(x) = \begin{cases} 0, & \text{if } x = 0 \\ \dfrac{60}{3-x}, & \text{if } 0 < x < 3 \\ 0, & \text{if } x = 3 \\ 5\sqrt{x}, & \text{if } x > 3 \end{cases}$$

where x is the input current (in milliamps) and $V(x)$ is the output voltage (in volts).

1. [A] Evaluate the limits.

$$\lim_{x\to 3^-} V(x) \qquad \lim_{x\to 3^+} V(x) \qquad \lim_{x\to 3} V(x)$$

2. [A+C] Is $V(x)$ continuous at $x = 3$? Why or why not?

3. [A+C] Graph $V(x)$. Label any discontinuities and describe what they mean in real-world terms?

4. [T] **Ethical Reasoning**

   (a) What are the physical and safety consequences of the spike at $x = 3$?

   (b) Why might a system designer allow a failure mode like this to remain instead of smoothing it out?

5. [A+C] Is $V(x)$ continuous on the following intervals?

   $$(0, 3), \quad [0, 3), \quad [0, 3], \quad (2, 5), \quad [3, 5), \quad (3, 5], \quad (5, +\infty), \quad [3, +\infty)$$

3 Calculus on Tour: Rate of Change – Climate Change in the Arctic

ACT for Ethical Reasoning

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Arctic sea ice reaches its minimum each September. On September 15, 2020, Arctic sea ice minimum set the second lowest on record, only after the record-low extent observed on September 17, 2012, according to the National Snow & Ice Data Center (NSIDC).

While reading news one day you notice that two newspapers are running headlines on the decline of Arctic sea ice extent.

The Daily Alarm

"Arctic Sea Ice Extent Declines By Over 20%!"

The Chill Out Weekly

"Arctic Sea Ice Extent Decreases By 4% A Year"

[C+T] Which of the two headlines do you think indicates a faster decrease of Arctic sea ice extent? Do they give you enough information to assess the rate at which the ice cap is disappearing?

Let's analyze the data and quantify change. Obtain the latest sea ice data from the NSIDC[2]. The Excel dataset sheet 1 contains monthly average Arctic sea ice extent for each year since 1979, in units of million square km.

[2] https://nsidc.org/arcticseaicenews/sea-ice-tools

1. [A+C] Plot the September data points from 1979 to 2024. Create a linear model by fitting a linear trendline to the data and display the equation. Write down the equation. What is the slope of the trendline? What does it represent in the context of Arctic sea ice extent?

2. [A+C] What projection would you make for the future of Arctic ice cap? What does your model predict for the Arctic sea ice extent in the year 2030 and 2050? When would the Arctic sea ice disappear?

3. [A] What is the percentage change and average rate of change of September Arctic sea ice extent from 2014 to 2024?

4. [A+C] What are the year-on-year percentage changes from 2014 to 2024? For example, 2014 to 2015, 2015 to 2016, and so on. When did the most dramatic change occur?

5. [C+T] Revisit the two headlines. Were they wrong? Based on your data analysis, can you come up with a headline that describes the decline of Arctic sea ice extent since 2014 more accurately?

6. [T] **Ethical Reflection:** How can clear and accurate communication about climate science influence public understanding and ethical decision-making?

4 Calculus on Tour: Derivatives – Affordable Housing

ACT for Ethical Reasoning

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A city is experiencing rapid economic growth and housing prices have been on the rise. The city council is debating whether to rezone downtown to permit the construction of high-end condominiums. Supporters argue that luxury development could generate more funding for city services, while critics warn that rising prices may displace long-time residents, especially low-income renters and first-time homebuyers. To inform the city's decision, you've been asked to analyze how housing demand changes with price, and what trade-offs the city may face. The housing demand is modeled by a **logistic price response function**, as previously introduced

$$y(x) = \frac{100}{1 + e^{0.6(x-4)}},$$

where x is the price (in \$100,000s) and $y(x)$ is the demand (in number of housing units).

1. [A+C] What is the demand for housing when $x = 3$, 4, and 5? What do you observe?

2. [A] Compute $y'(x)$ using the quotient rule and chain rule.

3. [C] What does $y'(x)$ tell you about the demand?

4. [A+C] Write down the revenue function $R(x)$.

5. [A] Compute $R'(x)$ using the product rule.

6. [A+C] Evaluate $R'(x)$ at $x = 3$, 4, and 5. At what price is revenue increasing or decreasing?

7. [C] Explain the trade-off: Why might revenue increase even when demand decreases?

8. [T] **Ethical Reasoning**

   (a) If the city chooses to price housing at $x = 5$ (i.e., \$500,000), what happens to demand? Who benefits and who is left out?

   (b) Use utilitarianism or virtue ethics to argue: Should the city increase revenue from high-end homes or subsidize affordable housing even if revenue falls?

   (c) What does your derivative-based analysis not capture about real-world housing fairness and affordability?

Reflection and Takeaway

This activity shows how calculus and mathematical models can guide pricing, but they don't settle questions of equity or social justice. Derivatives tell us what's changing; ethical reasoning tells us what matters.

5 Calculus on Tour: L'Hôpital's Rule – An Encore

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Let's revisit the **logistic price response function** introduced previously

$$f(x) = \frac{100}{1 + e^{0.1(x-200)}},$$

where x is the price (in dollars) and $f(x)$ is the demand (in thousands of tickets).

Revenue is the total amount of money generated by the sale of goods or services, calculated as price per unit times the number of units sold. A business might raise prices to increase revenue, but higher prices can reduce demand, thus decreasing revenue.

1. [A] Write down the revenue function $R(x)$ for ticket sales.

2. [A+C] Can we use L'Hôpital's Rule to compute $\lim_{x\to 100} R(x)$? Why or why not?

3. [A] Compute the limits $\lim_{x\to 100} R(x)$ and $\lim_{x\to 172} R(x)$.

4. [A+C] Can we use L'Hôpital's Rule to compute $\lim_{x\to\infty} R(x)$? Why or why not?

5. [A+C] What is $\lim_{x\to\infty} R(x)$? Describe what this means in real-world terms?

6. [T] **Ethical Reasoning**

Some average ticket prices by artist in 2024[3] are listed below.

Artist	**Average Ticket Price**
U2	$ 272
Taylor Swift	$ 239
Bruno Mars	$ 218
Eagles	$ 213
Beyoncé	$ 209
Lady Gaga	$ 172
P!NK	$ 152
Coldplay	$ 104
Maroon 5	$ 85
Imagine Dragons	$ 83

(a) Do you believe all artists are trying to maximize revenue from ticket sales? If not, what other goals might be guiding their pricing decisions?

(b) Should artists and promoters consider income inequality when setting prices for live events? Is it ethically acceptable to make concerts a luxury experience for those who can afford it? Should they prioritize revenue, fairness, or fan access?

(c) [Fun Bonus] Choose one artist from the table or link provided whose pricing surprises you. Research one factor (e.g., venue size, fan demographics, tour format, etc.) and explain how it might justify or challenge the average ticket price.

[3] https://studyfinds.org/concert-ticket-prices-by-the-minute-bruno-mars/

6 Calculus on Tour: Implicit Differentiation – Under the Hood

ACT for Ethical Reasoning

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A car manufacturer is studying the relationship between a vehicle's fuel efficiency (in miles per gallon, or mpg) and its speed (in miles per hour, or mph) under wind resistance and mechanical strain. The relationship between the speed x and fuel efficiency y is modeled with the equation

$$xy + (x - 70)^2 + 110\ln(y) = 2800.$$

The engineers want to find out how fuel efficiency changes with speed.

1. [A+C] What quantity would tell you how fuel efficiency changes with speed?

2. [A] Can you solve the equation easily for y? Why or why not?

3. [A] Use implicit differentiation to find dy/dx.

4. [A+C] Evaluate dy/dx at $(x, y) = (30, 28), (50, 40), (70, 34)$, and $(80, 29)$ and interpret the result: What does the sign and magnitude tell you about how fuel efficiency is changing?

5. [A] Can you sketch what the curve of y versus x might look like based on your data?

6. [T] **Ethical Reasoning**

   The model shows that fuel efficiency decreases with increasing speed when driving above a certain speed (e.g., 50 mph), which is in alignment with real data.

   (a) Should car manufacturers or navigation apps encourage drivers to reduce speed to conserve fuel and reduce emissions? What ethical responsibility do manufacturers or designers have in helping consumers make environmentally responsible choices?

   (b) Driving faster may reduce travel time but increases fuel consumption and emissions. From a utilitarian perspective, is it justifiable to allow high-speed driving for convenience even if it leads to greater environmental harm? What balance should be struck between individual freedom and collective responsibility?

7 Calculus on Tour: Curve Sketching – Tracking Health

ACT for Ethical Reasoning

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A doctor is monitoring a patient whose condition is deteriorating over time. Without any treatment, the patient's health status follows the model

$$f(t) = 100e^{-0.2t},$$

where t represents time in days and $f(t)$ represents health on a scale from 0 to 100. This reflects a natural decline in health if no intervention is made.

The doctor considers administering a drug. The drug improves the patient's condition over time, modeled by

$$d(t) = 100(1 - e^{-0.1t}).$$

However, the drug also has side effects, which build up linearly,

$$s(t) = 0.2t.$$

The overall patient health condition is modeled as

$$\begin{aligned} h(t) &= f(t) + d(t) - s(t) \\ &= 100e^{-0.2t} + 100(1 - e^{-0.1t}) - 0.2t. \end{aligned}$$

1. [A+C] Sketch $f(t), d(t)$, and $s(t)$ separately. Describe the functions in real-world terms.

2. [A+C] What is $h(0)$, $h(250)$, and $h(500)$? Interpret your results.

3. [A] Compute $h'(t)$ and $h''(t)$?

4. [A] Find the critical points.

   Hint: After you obtain an equation, try a substitution: $x = e^{-0.1t}$ to turn that equation into a quadratic equation, and you'll be able to apply the quadratic formula.

5. [A] Identify whether the critical point gives a relative maximum or minimum.

6. [A] On which intervals is the curve increasing, and on which intervals is it decreasing?

7. [A] Find the inflection points.

8. [A] On which intervals is the curve concave up, and on which intervals is it concave down?

9. [A+C] When does the patient's condition start to improve with the drug?

10. [A+C] Will the patient's condition start to deteriorate again due to the side effect of the drug?

11. [A] Sketch a curve representing the patient's condition.

12. [T] **Ethical Reasoning**

The drug improves the patient's condition in the short term, but its side effects accumulate. At some point, the side effects may outweigh the benefits of the drug.

(a) If the math shows a clear turning point in the patient's overall health, is it ethical to continue administering the drug past that point? What factors might justify continuing or stopping treatment?

(b) What ethical responsibility is there to weigh all the factors, especially when outcomes are uncertain or trade-offs involve quality of life?

(c) Should this information be presented as part of informed consent? How should mathematical models like this inform shared decision-making?

8 Calculus on Tour: Applied Max/Min – EthikDrug

ACT for Ethical Reasoning

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A German pharmaceutical company "EthikDrug" produces a life-saving drug. The cost of manufacturing each unit of the drug is \$10, and the current selling price is \$100 per unit. The demand for the drug is given by a function

$$D(x) = 500 - 2x,$$

where x is the price in dollars and $D(x)$ is the quantity demanded. The company wants to determine the price that maximizes their profit.

Recall that **revenue** is calculated as price per unit times the number of units sold. **Profit** is revenue minus total cost.

1. [A+C] Write down an equation for calculating the profit $P(x)$.

2. [A+C] Formulate an applied max/min problem:

   Maximize _____ over _____.

3. [A] Solve the problem.

4. [T] **Ethical Reasoning**: Now you've arrived at the optimal price of the drug. There are several ethical dilemmas to be discussed.

   (a) What are you optimizing and who are you optimizing it for?

   (b) Setting a high price for a life-saving drug may limit its accessibility to those who need it most. How can fairness be defined and balanced to ensure equitable access and the goal of maximizing overall well-being?

   (c) Pricing decision can lead to social and economic impacts. How does the pricing decision affect different socioeconomic groups? Are there any social responsibilities that the company should consider while making these decisions? Give some examples.

   (d) What factors can affect employee welfare, considering a finite pool of funding available to the company? How will the pricing decision impact the employees working in the company? For example, will the decision lead to job losses, reduced compensation, or any negative consequences for the employees?

9 Calculus on Tour: Backstage Access – Sequences and Series

ACT for Ethical Reasoning

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A streaming company offers a music subscription for \$10/month. User listening behavior and company earnings are modeled over time. Suppose the number of new songs the average user listens to in month n is modeled by a sequence

$$a_n = 50 \cdot 0.8^{n-1}, \quad n = 1, 2, 3, \ldots$$

The company wants to estimate how many new songs the user will engage with over the first year, as well as long-term. This insight could affect how to design recommendation systems or how to justify subscription renewals and pricing.

1. [A+C] What type of sequence is this? What's the behavior of the sequence? What does this imply about user engagement over time?

2. [A+C] Find the first 3 terms of the sequence. What do they represent?

3. [A+C] Find the total number of new songs the user listens to in the first 12 months. Use the summation formula for geometric series.

4. [A+C] Does the series converge? If so, what's the total number of songs a user will ever listen to? Interpret this.

5. [A+C] What's the revenue collected from the user after the first 12 months?

6. [A+C] What's the average cost per song in month n? What about cumulative (long-term) cost per song?

7. [T] **Ethical Reasoning**

   (a) From an ethical standpoint, is it fair for a company to automatically renew a user's subscription if their engagement is clearly declining? Should companies take steps to notify users, offer alternatives, or adjust billing in light of reduced usage?

   (b) How could a company use this mathematical model to support sustainable revenue while also treating customers fairly? Should companies provide transparency about usage trends so that customers can make informed choices?

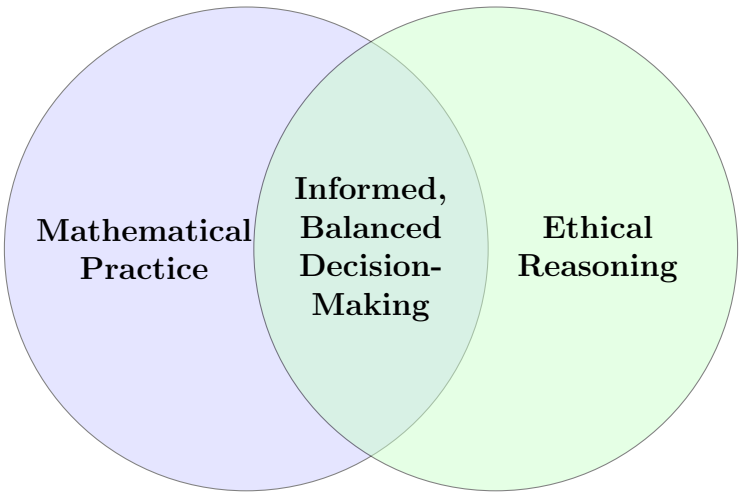
10 Calculus on Tour: Final ACT – Reflection and Takeaway

Throughout this calculus tour, we've explored how tools like limits, derivatives, and optimization can be applied to real-world problems, from maximizing profit to managing health outcomes. These mathematical techniques offer clarity and predictive power, enabling us to model complex systems and make informed decisions.

However, the role of calculus doesn't end at computation. The deeper challenge lies in interpreting these results within an ethical framework. Whether the context involves business, government, healthcare, or environmental sustainability, decision-makers must weigh financial goals against social responsibility, fairness, and long-term consequences.

Calculus, then, becomes not just a skill to acquire, but a lens through which we can balance competing priorities. Ethical reasoning should guide how we use mathematical models and methods to ensure that our actions promote equity and accessibility. Effective strategies may include involving stakeholders, considering alternative ways and outcomes, and aligning our goals with core ethical values.

Ultimately, this tour shows that calculus is not only a powerful analytical tool, but also, when paired with ethical reflection, a means for making more responsible decisions in a complex world.



Final Reflection: As you look back on this calculus tour, consider not just the calculus tools you've learned, but also the ethical decisions they can inform. How might your answer change depending on whether you're thinking as a business owner, a policymaker, or a citizen? Whether a model points toward maximizing profit or minimizing time, what values or voices might be left out? Going forward, how can you use both calculus and ethical reasoning to make decisions that are not only effective, but also responsible?