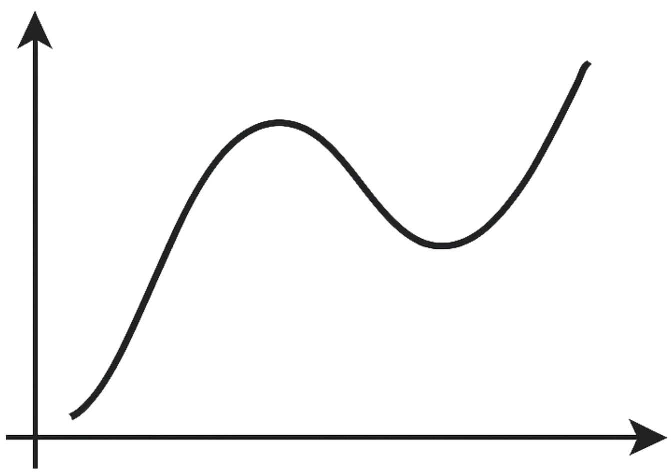
Differential Equations

Modeling a Changing World

ETHICAL REASONING COMPASS



Mathematics Gives Direction
Ethical Reasoning Guides the Path

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Welcome!

This journey invites you to explore differential equations not just as a mathematical tool, but as a way of understanding how the world changes, and how our choices shape that change. From modeling disease outbreaks and drug dosages to predicting climate trends and managing natural resources, differential equations help us describe dynamic systems and anticipate future outcomes. But solving the math is only part of the challenge. At each stop on this journey, you'll interpret models, weigh competing goals, and reflect on how ethical reasoning can guide responsible decision-making. Whether you're tracking population, tuning a pacemaker, or planning for sustainability, remember: mathematics gives us direction – ethical reasoning guides the path.

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1 Opening ACT: Why Model Change?

ACT for Ethical Reasoning

- **A**nalyze – Do the calculations: compute, graph, solve.
- **C**onnect – Interpret results in context: what do they mean in the real world?
- **T**hink Critically – Reason ethically: evaluate trade-offs, justify decisions, consider broader consequences.

These three components (A, C, and T) can occur in different orders and iterations. Sometimes, you need to Think Critically (T) before you Analyze (A)!

Differential equations are about change: How do populations grow? How do resources get used up over time? How does climate change? How do diseases spread? Unlike static equations, differential equations model dynamic systems and describe how something evolves and what might happen next.

But just because we can model a system doesn't mean we know what to do with it. For example, a mathematical model can show how fast a natural resource is depleting but not whether it's fair to keep extracting it.

Your job is not just to construct or solve differential equations, but to understand the systems behind them and the choices they raise. You will use differential equations to describe what's happening, and ethical reasoning to ask what should happen.

1. [A+C] What is one real-world example you think could be modeled by a differential equation? What quantity would change, and what might cause the change?

2. [A+C] Why do you think it's useful to model change over time instead of just using a static equation? Who might rely on these kinds of models?

3. [T] What responsibilities do scientists, policymakers, and businesses have when acting on the results of mathematical models? Can we rely on the math alone, or must decisions also include something more?

2 First-Order Linear ODEs: Are You Saving Enough?

ACT for Ethical Reasoning

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Suppose you deposit money into a savings account that earns interest. While the account balance grows with interest, inflation decreases the real value of that money. Inflation is an increase in the general level of prices for goods and services, thus reducing the purchasing power of money.

Let $y(t)$ represent the real value (in today's dollars) of the savings account at time t (in years). Suppose the account earns 4% interest per year, but inflation reduces the value at 2% per year. The net effect can be modeled by a first-order linear differential equation

$$\frac{dy}{dt} = 0.04y - 0.02y = 0.02y$$

1. [A] Solve the equation using separation of variables.

2. [A+C] Suppose you start with y(0)=1000. What is the real value of the savings after 10 years, 20 years, 30 years?

3. [A+C] What about in the long run? Sketch the solution. What assumption does this rely on?

4. [A+C] Suppose the inflation rate increases to 4% while interest remains at 4%. Write the new equation. What does this imply about the future value of savings? Plot the solution.

5. [A+C] What if the inflation rate increases to 5%? Sketch the solution. Compare your solution curves.

6. [T] **Ethical Reasoning**

   (a) Is it ethical to promote savings if inflation is outpacing interest? What trade-offs would that involve?

   (b) How can models like this inform financial planning?

7. [A+C+T] Suppose you make a regular deposit of \$500 per year. The differential equation becomes

$$\frac{dy}{dt} = 0.02y + 500.$$

Solve this new equation and discuss how it changes the financial outlook over time. Also compare scenarios with net interest rates of 0% and −1%.

3 Logistic Model: Spread of Misinformation

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A false news begins to spread on social media. At first, only a few people see it. As more people encounter and share it, the spread grows quickly. But eventually, the rate slows down, because most people may have seen it or fact-checked it, or social media platforms begin to intervene. This process can be described using a **logistic model**

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right),$$

where $y(t)$ represent the number of people (in thousands) who have seen the false news at time t (in hours). The parameter r is the spread rate of the misinformation and K is the maximum exposure, a.k.a, the carrying capacity, $r > 0, K > 0$. The logistic differential equation models scenarios where the population grows quickly at first, but the growth slows down as it approaches a saturation point.

1. [A+C] Suppose $r = 0.7$ and $K = 100$ (thousand). Write down the differential equation. Find the equilibrium solutions and determine their stability. What do they mean in context?

2. [A+C] Starting at $y(0) = 1$, what happens over time?

3. [A+C] At what point does the spread rate slow down?

4. [A] Sketch the slope fields and solution graph starting at $y(0) = 1$.

5. [A] Solve the differential equation using the method of separation of variable.

6. [T] **Ethical Reasoning**

   (a) If a social media platform knows a piece of content is false, when is the best time to intervene?

   (b) What ethical responsibilities do platforms or individuals have in using models like this to flag or suppress misinformation?

   (c) Can or should one use the same model to promote helpful or truthful content? Explain.

4 Logistic Model with Harvesting: Growth vs. Extraction

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A town extracts a natural resource, such as groundwater, timber, or a rare mineral, at a rate proportional to the current amount. The resource also naturally replenishes over time following a logistic model introduced previously.

Let $y(t)$ represent the amount of the resource at time t. The resource level is modeled by the differential equation

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right) - hy,$$

where r is the natural replenishment rate, K is the carrying capacity, and h is the extraction rate coefficient, $r > 0, K > 0, h \geq 0$. This is a **logistic model with harvesting**.

1. [A] Suppose $r = 1$ and $K = 1000$. Write down the differential equation. Find the equilibrium solutions (in terms of h).

2. [A+C] Determine the value of h for which the natural resource is eventually depleted.

3. [A] Plot the solution graphs for $h = 0.4$ and 0.8, respectively.

4. [A+C] What happens if the initial resource is above or below the equilibrium? What is the long-term behavior of the resource?

5. [T] **Ethical Reasoning**

   (a) Suppose it's argued that extracting at $h = 0.8$ is economically optimal. As a technical advisor, how might you use both math and ethical reasoning to critique or support this policy?

   (b) How might a mathematical model like this inform public policy or community, who should decide, and what are the limitations of relying solely on such models?

5 Warming Planet: Modeling Climate Change

ACT for Ethical Reasoning

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Carbon dioxide (CO_2) is the largest contributor of greenhouse gases. Human activities are emitting carbon dioxide into the atmosphere faster than natural processes can take it out. The CO_2 levels today are higher than at any point in at least the past 800,000 years.

The Paris Climate Agreement, adopted by 196 countries on 12 December 2015, represented a huge historic step in re-imagining a fossil-free future for our planet. The treaty called for keeping global temperature rise well below 2°C relative to pre-industrial levels by 2100, as well as agreeing to pursue efforts to limit the increase to 1.5°C above pre-industrial levels.

Differential equations can be used to model how global temperature responds to CO_2 buildup and how natural systems try to restore balance. The rate of temperature change is viewed as a balance between heating from CO_2 and cooling by Earth's natural feedback systems,

$$\frac{dT}{dt} = a - bT,$$

where $T(t)$ represent the global temperature anomaly (in °C above pre-industrial baseline) at time t (in decades). The parameter a is the warming rate due to CO_2 and b is the rate at which Earth dissipates heat, $a > 0, b > 0$.

1. [A+C] What are the units of a and b?

2. [A+C] Suppose $a = 0.4$ and $b = 0.2$, solve the differential equation for $T(0) = 0$ (pre-industrial baseline).

3. [A+C] Sketch the solution graph. What's the behavior of the temperature over time? What does that mean for climate policy?

4. [T] **Ethical Reasoning**

   (a) If you're using this model to help evaluate climate targets, how do you balance urgency, uncertainty, and socioeconomic impact?

   (b) Who should bear the cost of reducing emissions? Should it be equally distributed across countries, industries, or generations?

   (c) Who is most likely to suffer from the consequences of climate change, and are those impacts distributed equally? What principles of fairness or responsibility should guide global climate policy?

We modify the previous model by introducing emissions reduction over time, thanks to global efforts to reduce CO_2 emissions. Let r be the rate of emissions reduction and the differential equation becomes

$$\frac{dT}{dt} = a - rt - bT.$$

This is a non-autonomous, first-order linear ODE.

2. [A+C] What are the units of r?

3. [A+C] Suppose $a = 0.4$, $b = 0.2$, and $r = 0.02$, solve the differential equation for $T(0) = 1.5$ (°C above pre-industrial baseline) using the method of integrating factors.

4. [A+C] Sketch the solution graph. At what time does warming stop increasing? What happens after that?

5. [A+C+T] What does your analysis suggest about the timing of intervention? How does earlier or faster emissions reduction change the temperature outcome? From an ethical standpoint, how do delayed emissions reductions impact future generations?

6 Second-Order Linear ODEs: A Healthy Rhythm

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Did you know your heart acts like a biological oscillator? With each beat, it contracts and relaxes in a rhythmic cycle, continuously pumping blood throughout the body. After each contraction, the heart muscle gradually returns to its resting state, a process that can be modeled using a second-order differential equation. This type of modeling is especially important in the design of pacemakers, which must respond to and regulate the heart's natural oscillations to maintain a healthy rhythm.

We model the vertical displacement of the heart wall using the second-order differential equation

$$my'' + by' + ky = 0,$$

where $y(t)$ is the deviation from resting position (in millimeters) at time t (in seconds), m is an effective mass (in kilograms), relating the biomechanical model of tissue, b is the damping coefficient that models internal resistance such as tissue friction, and k reflects the restoring force due to elasticity of the heart muscle.

1. [A+C] What are the units for b and k? Suppose $m = 1, b = 2$, and $k = 10$, write the differential equation.

2. [A] Find the characteristic equation and classify the system as underdamped, overdamped, critically damped, or undamped.

3. [C] What does this tell you about how the heart tissue returns to rest?

4. [A+C] Sketch a graph (qualitatively) of the general solution. Describe what it tells you about the rhythm of the heartbeat.

5. [A+C] What would happen if b were too low? Too high? How would this affect outcomes?

6. [A+C] Solve the equation for a heart at rest with no initial displacement or velocity.

7. [T] **Ethical Reasoning**

   (a) If you use this model to help doctors design pacemaker settings, how can it guide safe and effective programming? What trade-offs must be considered (e.g., risks, costs, and uncertainties) when applying the model in real clinical settings?

   (b) What ethical responsibilities arise when applying mathematical models to real patient care? How should one balance mathematical precision with human variability, and when should medical judgment override what the model predicts?

7 System of ODEs: Modeling Infectious Disease

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Suppose the dynamics of an infectious disease within a population in a resource-limited environment are modeled by a system of differential equations

$$S' = S[1 - (S + I)] - aSI$$
$$I' = I[1 - (S + I)] + aSI - bI$$

where $S(t)$ is the percentage of susceptible population at time t, $I(t)$ is the percentage of infected population, $0 \leq S + I \leq 1$.

The parameter a is the rate of infection, representing how quickly the susceptible becomes infected through contact with infected individuals, and b is the mortality rate caused by the disease, representing how quickly infected individuals die due to the disease, $a > 0$ and $b > 0$.

1. [A] Determine the equilibrium solutions of the system, thus the long-term outcome for the population.

2. [A] For $a = 0.8$, find the equilibrium solutions for $b = 0.2, 0.4$, and 0.6, respectively.

3. [A+C] Which of these values of b results in the lowest long-term total population? Explain why this occurs.

4. [T] **Ethical Reasoning**

   (a) Which do you think is worse: a disease with a mortality rate of 0.2 or one with a mortality rate of 0.6?

   (b) This question aims to unpack what "worse" means, and for whom? From a utilitarian perspective, which mortality rate, $b = 0.2$ or $b = 0.6$, would be preferred, and why? How does this reflect the goal of maximizing overall well-being?

   (c) Now consider the perspective of an individual infected by the disease: which mortality rate would they prefer, and why? What ethical tensions or dilemmas might arise?

   (d) What additional complexities should be incorporated to make the model more practical, and who should be responsible for its feasibility?

8 System of ODEs: Tracking Pandemics

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At the end of 2019, a novel coronavirus emerged, later named COVID-19, and rapidly spread across the globe. It became the seventh known member of the coronavirus family capable of infecting humans, joining MERS-CoV and SARS-CoV. Just three months after its discovery, the World Health Organization (WHO) officially declared the outbreak a pandemic.

The term pandemic originates from the Greek word 'pandemos', meaning "pertaining to all people". It is a combination of pan, meaning "all", and 'demos' means "people". Unlike an epidemic, which is confined to a specific city, region, or country, a pandemic spreads across national borders and can affect populations worldwide. Increased global travel and mobility have significantly heightened the risk of new infectious diseases spreading.

Infectious disease modeling plays a crucial role in controlling outbreaks. Well-designed models not only help predict the likely trajectory of an epidemic, but also identify the most effective and feasible strategies for containing it.

The simplest model is the **SIR model** (Susceptible-Infected-Recovered). It makes several basic assumptions:

A1. The total population is constant.

A2. Once an individual has been infected and subsequently recovered, that individual cannot be re-infected, such as in the case of measles, mumps, and smallpox.

A3. The rate of transmission of the disease is proportional to the number of encounters between susceptible and infected individuals.

A4. The rate at which infected individuals recover is proportional to the number of infected.

Let $S(t)$ be the fraction of susceptible individuals at time t, $I(t)$ the fraction of infected individuals, and $R(t)$ the fraction of recovered individuals. $S(t) + I(t) + R(t) = 1$. Let a be the transmission rate and r the recovery rate, $a > 0$ and $r > 0$. The SIR model is illustrated schematically in Figure 1 and described by the following nonlinear system of ODEs.

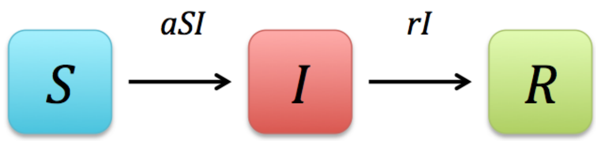


Figure 1: The SIR model.

$$S'(t) = -aSI \quad (1)$$
$$I'(t) = aSI - rI \quad (2)$$
$$R'(t) = rI \quad (3)$$

1. [A] Consider the initial population fractions $S(0) = 0.99, I(0) = 0.01, R(0) = 0$ ($S + I + R = 1$), $a = 1, r = 0.2$, and a time period of 30 days. In GeoGebra, solve the nonlinear system (using 'NSolveODE'), plot the solutions, and visualize the interaction of the three population groups.

   An example demonstration is provided[1] with a screenshot in Figure 2. Adjust the parameters values to examine different behaviors and view the changes in action. What do you observe by increasing or decreasing a? Increasing or decreasing r?

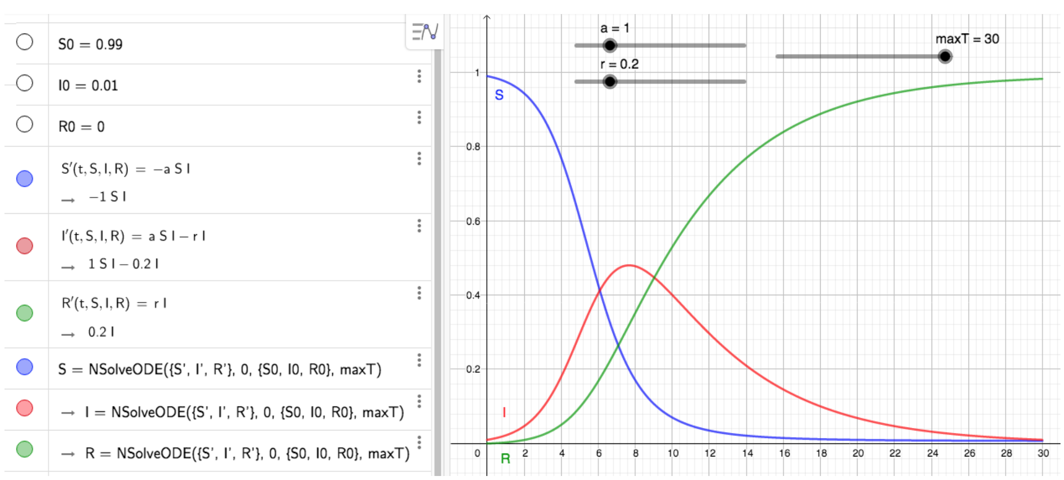


Figure 2: A screenshot of GeoGebra dynamic constructions.

[1] https://www.geogebra.org/classic/zumeuffq, created by Jue Wang.

The *basic reproduction number* R_0, "R naught", is introduced to measure the transmission potential of a disease. It is the expected number of secondary infections produced by a single infective. For example, if the R_0 for measles in a population is 16, then we would expect each new case of measles to produce 16 new secondary cases over the period of time during which the infected individual can actually spread the disease. R_0 is essentially a metric of how contagious a disease is.

R_0 is affected by several factors: the rate of contacts in the host population, the probability of infection being transmitted during contact, and the duration of infectiousness. In the SIR model,

$$R_0 = \frac{a}{r}. \tag{4}$$

2. [A+C] According to the differential equation for dI/dt, under what conditions will $I(t)$ be increasing? decreasing? What does this mean about the spread of the disease? Using this result, explain whether quarantine will be effective against the disease.

3. [A] Use the chain rule to show that

$$\frac{dI}{dS} = -1 + \frac{1}{R_0 S} \tag{5}$$

Two features of this new equation are particularly worth noting:

- The only parameter that appears is R_0, the reproduction number.
- The equation is independent of time. That is, what we learn about the relationship between S and I must be true for the entire duration of the epidemic.

4. [A+C] Compute d^2I/dS^2. Determine when the number of infected will begin to decrease. Compare this to your result from 2.

5. [A] Show that I must have the form

$$I = -S + \frac{1}{R_0} \ln S + C \tag{6}$$

where C is a constant.

6. [A+C] For a disease such as COVID-19, $I(0)$ is approximately 0 and $S(0)$ is approximately 1. A long time after the onset of the epidemic, we have $I(\infty)$ approximately 0 again, and $S(\infty)$ has settled to its steady state value, observable as the fraction of the population that did not get the disease. Explain why

$$R_0 = \frac{\ln S_\infty}{S_\infty - 1}.$$

7. [C] Describe the meaning and significance of "herd immunity". How can vaccination lead to herd immunity?

8. [A+C] Recovered individuals may lose immunity and become re-infected, as is possible with diseases like COVID-19. In reality, the disease may also lead to deaths. You are encouraged to explore these possibilities further by modifying the previous diagram and developing new models.

9. [T] **Ethical Reasoning**

   (a) If you're using this model to help governments or hospitals prepare for an outbreak, how should you balance mathematical predictions, uncertainty, transparency, and public communication?

   (b) The SIR model assumes that the entire population mixes uniformly, i.e., everyone has equal chance of contact. In reality, however, some communities may be more vulnerable due to housing, income, or access to healthcare. What are the ethical consequences of relying on simplified models that may ignore social inequality?

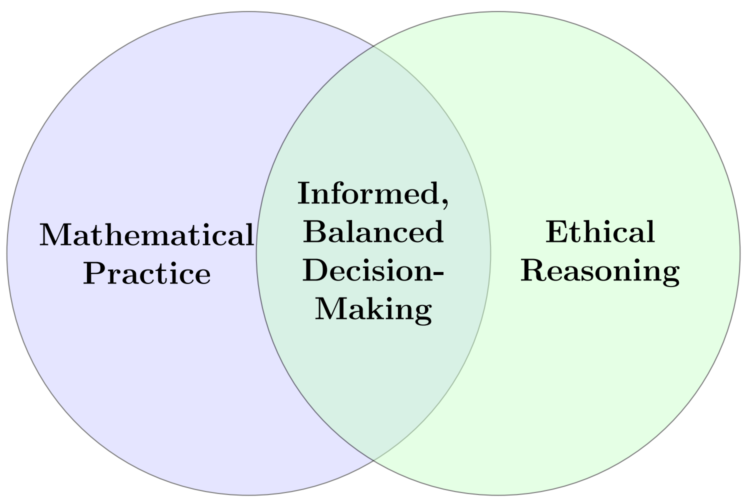
9 Final ACT: Reflection and Takeaway

We've explored how differential equations can be applied to real-world problems, from modeling climate change to tracking pandemics. These mathematical techniques offer clarity and predictive power, enabling us to model complex systems and make informed decisions.

However, the role of differential equations doesn't end at computation. The deeper challenge lies in interpreting these results within an ethical framework. Whether the context involves business, government, healthcare, or environmental sustainability, decision-makers must weigh financial goals against social responsibility, fairness, and long-term consequences.

Differential equation techniques, then, become not just a skill to acquire, but a lens through which we can balance competing priorities. Ethical reasoning should guide how we use mathematical models and methods to ensure that our actions promote equity and accessibility. Effective strategies may include involving stakeholders, considering alternative ways and outcomes, and aligning our goals with core ethical values.

Ultimately, modeling with differential equation not only offers a powerful analytical tool, but also, when paired with ethical reflection, a means for making more responsible decisions in a complex world.



Final Reflection: As you look back on the modeling process, consider not just the differential equations you've learned, but also the ethical decisions they can inform. How might your answer change depending on whether you're thinking as a business owner, a policymaker, or a citizen? Whether a model points toward maximizing profit or minimizing time, what values or voices might be left out? Going forward, how can you use both differential equations and ethical reasoning to make decisions that are not only effective, but also responsible?